Cauchy's Root Test :-

The Series $\sum u_n$ of positive terms is ~~Convergent~~

(i) Convergent if $u_n^{\frac{1}{n}} < 1$ for all $n \geq m$

(ii) divergent if $u_n^{1/n} \geq 1$ for all $n \geq m$.

Proof :- we can assume.

$$u_n^{1/n} < r$$

$$\Rightarrow u_n < r^n, \quad \forall n, \; r<1$$

$$\therefore S_n = \sum u_n < 1 + r^2 + r^3 + \cdots + r^n$$

$$= \frac{r(1-r^n)}{(1-r)} < \frac{r}{1-r}$$

$$= K$$

$$\therefore S_n < K, \quad \forall n$$

Hence the sequence is bounded and monotonic increasing, so, it is convergent

$\therefore \sum u_n$ is Convergent

Similary, we can show.

It is divergent if $u_n^{1/n} \geq 1$.

Example : Test the Convergency of Series

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots$$

$$u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$$

$$u_n^{1/n} = \left(\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right)^{-1}$$

$$= \left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^{n} - 1\right]^{-1}$$

$$= \left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^{n} - 1\right]^{-1}$$

Hence, $\lim_{n\to\infty} u_n^{1/n} = 1 \cdot (e-1)^{-1}$ $\quad \left\{ \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \right.$

$$= \frac{1}{e-1} < 1, \text{ Since } e > 2$$

Hence, By Cauchy's root test, the Series is Convergent.